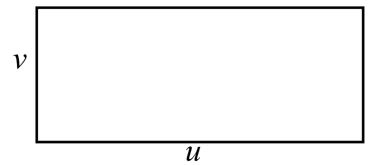
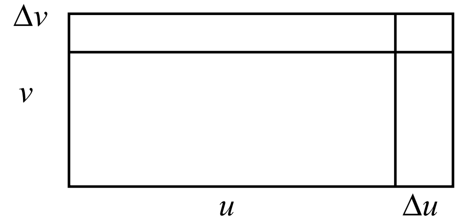
# Product Rule

Let $p(x) = u(x)\cdot v(x)$, where $u$ and $v$ are both differentiable functions. The product, $p$, may be represented by the area of the rectangle.




Now, assume that $u$ and $v$ increase by amounts $\Delta u$ and $\Delta v$, respectively.




From our picture, we find

$$\Delta p = u\cdot\Delta v + v\cdot\Delta u + \Delta u\cdot\Delta v$$

Now divide by $\Delta x$.

$$\frac{\Delta p}{\Delta x} = u\cdot\frac{\Delta v}{\Delta x} + v\cdot\frac{\Delta u}{\Delta x} + \frac{\Delta u\cdot\Delta v}{\Delta x}$$

For the derivative, we need to consider the limit as $\Delta x \to 0$.

$$\begin{aligned}\frac{dp}{dx} = \lim_{\Delta x\to 0}\frac{\Delta p}{\Delta x} &= u\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} + v\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\frac{\Delta u\cdot\Delta v}{\Delta x}\\ &= u\cdot\frac{dv}{dx} \quad + \quad v\cdot\frac{du}{dx} \quad + \quad \lim_{\Delta x\to 0}\Delta u\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\end{aligned}$$

It's that last term which needs to politely disappear. Fortunately, that's not difficult. Since $u$ is differentiable, it is also continuous. This implies that $\lim\limits_{\Delta x\to 0}\Delta u = 0$, so the last term will in fact be 0.

Alternatively, consider

$$\lim_{\Delta x\to 0}\frac{\Delta u\cdot\Delta v}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\cdot\frac{\Delta v}{\Delta x}\cdot\Delta x = \frac{du}{dx}\cdot\frac{dv}{dx}\cdot\lim_{\Delta x\to 0}\Delta x = 0$$

This gives us the desired formula,

$$\frac{dp}{dx} = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx} \qquad \text{(Yea!)}$$

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