So What's a Derivative?

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References:

CCH → Calculus, Deborah Hughes-Hallett, Andrew Gleason, et al., Wiley.
OZ → Calculus from Graphical, Numerical, and Symbolic Points of View, Arnold Ostebee and Paul Zorn.
Foerster → Calculus: Concepts and Applications, Paul Foerster, Key Curriculum Press.
Stewart → Calculus: Concepts and Contexts, James Stewart.

(1) OZ P. 70, 2nd ed. Interpret each sentence below as a statement about a function and its derivatives. In each case, indicate clearly what the function is and what each symbol means. [Note: Assuming first and second derivatives.]
(a) The child's temperature is still rising, but the penicillin seems to be taking effect.
(b) The cost of a new car is increasing at an increasing rate.

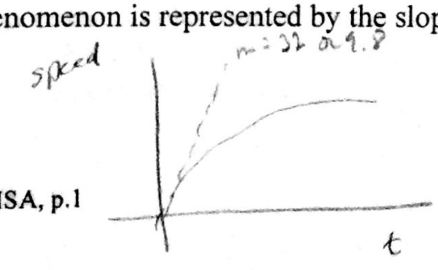
(2) CCH P. 94,5th ed. The cost of extracting T tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say that $f'(2000) = 100$?

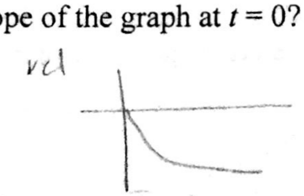
(3) You've just baked a pizza and you take it out of a 425° F oven. Then the phone rings and the pizza sits cooling on the counter. Let k represent the temperature of the pizza t minutes after it is taken from the oven.
What are the units of k'?
What does $k'(8) = -2$ mean in terms of pizza? (Be very specific to the situation.)
Is $k'(2) < k'(8)$? Explain.

yes
larger negative

(4) CCH P. 97:23 (a) If you jump out of an airplane without a parachute, you will fall faster and faster until wind resistance causes you to approach a steady velocity, called a terminal velocity. Sketch a graph of your velocity against time.
(b) Explain the concavity of your graph.
(c) Assuming wind resistance to be negligible at $t = 0$, what natural phenomenon is represented by the slope of the graph at $t = 0$?

slope = acceleration / gravity
initially, free fall,
then air resistance kicks in





(5) CCH P. 103:24, 5th ed. "Winning the war on poverty" has been described cynically as slowing the rate at which people are slipping below the poverty line. Assuming that this is happening:

(a) Sketch a graph of the total number of people in poverty against time.

(b) If N is the number of people below the poverty line at time t, what are the signs of $\dfrac{dN}{dt}$ and $\dfrac{d^2N}{dt^2}$?

pos neg

(6) CCH P. 130:7 (1st ed) IBM-Peru uses second derivatives to assess the relative success of various advertising campaigns. They assume that all campaigns produce some increase in sales. If a graph of sales against time shows a positive second derivative during a new advertising campaign, what does this suggest to IBM management? Why? What does a negative second derivative during a campaign suggest?

(7) CCH P. 103:23, 5th ed. In economics, *total utility* refers to the total satisfaction from consuming some commodity. According to the economist Paul Samuelson:

> As you consume more of the same good, the total (psychological) utility increases. However...with successive new units of the good, your total utility will grow at a slower and slower rate because of a fundamental tendency for your psychological ability to appreciate more of the good to become less keen.

(a) Sketch the total utility as a function of the number of units consumed.

(b) In terms of derivatives, what is Samuelson saying?

(8) Foerster P. 1:17, 2nd ed. A pebble is stuck in the tread of a car tire. As the wheel turns, the distance, y inches, between the pebble and the road at various times, t seconds, is given in the chart.

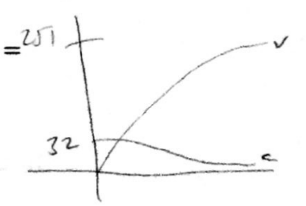
t sec	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y in	.63	.54	.45	.34	.22	.00	.22	.34	.45

(a) About how fast is y changing at each time? $t = 1.4$, $t = 1.7$, $t = 1.9$

(b) At what time does the stone strike the pavement? Justify your answer.

(9) Foerster P. 103: 8 Phoebe jumps from an airplane. While she free-falls, her downward velocity, $v(t)$ ft/sec, as a function of t seconds since the jump, is $v(t) = 251(1 - 0.88^t)$.

vel down



(a) Plot the velocity, v, and acceleration, a, on the same screen. Use an x-window (actually a t-window) of 0 sec to 30 sec. Sketch the results.

(b) What is Phoebe's acceleration when she first jumps? Why do you suppose the acceleration decreases as she moves faster and faster? $a = 32$

(c) What does the limit of $v(t)$ seem to be as t approaches infinity? This limit is called the terminal velocity. 251

(10) Stewart P.143: 16, 4th ed. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.

(a) Find the average velocities over the following time intervals:

(I) [3, 4] (ii) [3.5, 4] (iii) [4, 5] (iv) [4, 4.5]

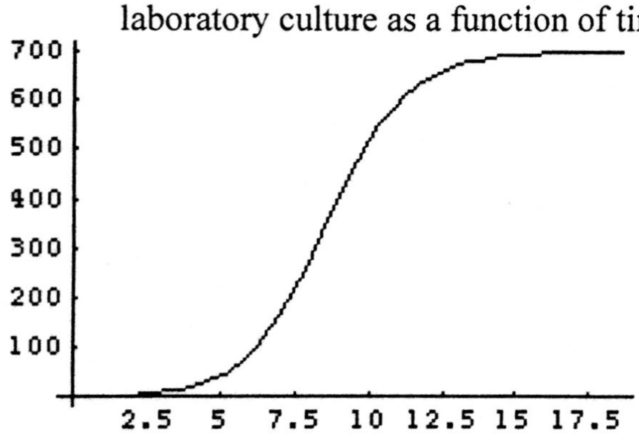
(b) Find the instantaneous velocity when $t = 4$?

(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

(11) Stewart P. 144: 46, 4th ed. If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in 1 h, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$V(t) = 100{,}000\left(1 - \frac{t}{60}\right)^2$ for $0 \le t \le 60$. Find the rate at which the water is flowing out of the tank as a function of t. What are its units? For times $t = 0$, 10, 20, 30, 40, 50, and 60, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

(12) Stewart P.162:10, 4th ed. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.



(a) Describe how the rate of population increase varies.

(b) When is this rate highest?

(c) On what intervals is the population function concave upward or downward?

(d) Estimate the coordinates of the inflection point.

(13) Let f be a continuous function such that $f(2) = -3$, $f'(2) = 5$, and $f''(x) < -2$ for all x. Approximate $f(2.2)$. Is this approximation too high or too low? Why?

$\downarrow$ conc down

$\Rightarrow$ approx too hi

$y + 3 = 5(x - 2)$

$y = 5x - 13$

Stewart P. 150: 9 Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or downward? Why?

↑ larger

conc down

learn more at first, then less

(O/Z P. 140: 62, 2nd ed.) Let $f(x) = 2^x$ and $g(x) = 7x/3 + 1$. What is the slope of the line tangent to the curve $y = f(x)$ at the point in [0, 3] where the vertical distance between f and g is the greatest?